

Let's talk maths: A model for teaching to reveal student understandings

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'Miss, I don't understand!'

'What don't you understand?'

'All of it!'

Teachers are often faced with the difficulty of deciding exactly what problems the students are experiencing, and the extent to which students might understand 'some of it'. The one-on-one approach to helping students is most common, but very time consuming, and is an approach that may only serve to fix an immediate difficulty, perhaps by addressing procedural aspects rather than the concepts. One reason for this may be the very real difficulty both parties have in identifying exactly what the problem(s) could be.

Teachers questioning students about their mathematical understanding often results in a conversation with the teacher leading in ways that he or she feels appropriate and the student responding in the accepted form. However, as Barnes (1999) reveals in the recorded observations of student talk in small groups, student-student talk is much freer. The tenor of the talk changes when the teacher is present but it also appears to remain more free-ranging than would appear to be the case in other classroom situations. It is this free-ranging nature of student-student conversations that can reveal much about students' understandings. The model described in this paper utilises a situation where students can engage in a conversation with each other, with the teacher present. The teacher partly 'eavesdrops' and partly guides the conversation. The guidance is not directed towards some goal determined by the teacher (such as having the students solve a problem), but towards the teacher finding out what is going on in the students' thinking.

The underlying premise for the model is that mathematics is a language (e.g., Pimm, 1987). Techniques of language-arts teaching together with non-routine tasks generate student conversations, which provide linguistic clues to student understanding. In this article, the theoretical bases for the model are briefly outlined and two snapshots of a discussion in a class conducted with a small group of students will be presented as examples.

An outline of the theoretical bases of the model

There are three major aspects which form the bases of the teaching and learning model discussed in this article. They are: the employment of language arts techniques; the use of linguistic structures as indicators of student understanding; and the use of non-routine questions to promote conversation.

The employment of language arts techniques

If mathematics is viewed as a language (e.g., Pimm, 1987), then strategies associated with the teaching of the language arts may be adapted to teach mathematics. Bickmore-Brand (1993) identifies such strategies as immersing students in the environment of the language, encouraging the use of the language in appropriate contexts, and careful modelling of the language by teachers. In the example presented in this paper the context and interest are established by the use of non-routine questions. The immersion is achieved by encouraging discussion between the students and the teacher whilst the teacher models appropriate language skills and encourages students to use clear, precise and meaningful language.

The language-based learning environment is based on a model outlined by Bell (1993) developed for teaching mathematics to ESL (English as a Second Language) students. However, it is a model that is also particularly useful when working with students whose mathematical experiences have not led to robust feelings of success. The model consists of five instructional elements and five conversational elements. The elements are:

1. a thematic focus chosen by the teacher;
2. the use of relevant student backgrounds and knowledge;
3. direct teaching when necessary;
4. the modelling and promotion of appropriate and technical language;
5. the asking of questions such as: 'How do we know...?', 'Show me why...?';
6. the use of few 'known answer' questions;
7. acknowledging and responding to all student contributions;
8. a discourse that builds on and connects ideas;
9. a challenging, non-threatening environment;
10. a general participation in a conversation that takes the direction dictated by the students rather than the teacher (Bell, 1993).

This framework establishes the environment that encourages the conversations that provide linguistic clues to student understandings.

The use of linguistic structures as indicators of student understanding

Research by Bills and Grey (2001) suggests that students' use of language may indicate the extent of their mathematical understanding. Bills and Grey (2001) examined the responses of young students who, after performing a mental calculation were asked to describe what was in their head as they carried out the calculation. Their responses were categorised as *particular* when students simply described the calculation with specific numbers, *generic*

when the numbers were used 'as a vehicle to describe a procedure', and *general* when the students made little or no reference to specific numbers as they described the procedure. In the discussion of the conversations considered in this paper, the terms 'particular', 'generic' and 'general' are used in the context of algebra (generalised arithmetic) and geometry.

The learning and teaching model outlined emphasises the verbal aspects of mathematics communication in contrast to the more common emphasis on the written forms. For students lacking confidence, speaking about their ideas may often be easier than writing (McGregor, 1993). Writing takes more time than speaking. Ideas can be lost as students struggle to find the right words, or, in the case of mathematics, the right symbols. The struggle to conceptualise mathematics does not appear so obvious when students speak about their mathematics, particularly when the use of informal language is acceptable and the talk is with peers. Student thoughts are made public, and the group can collectively assist with appropriate vocabulary. Students often interpret one another's faltering attempts in ways that appear mysterious to the teacher and so act as mediators with the teacher. The term 'conversation' is used in this paper to convey a degree of equality between the participants (teacher included), the informality of the classroom and the free-ranging nature of the course of the conversations. As students come to realise that they need more precise ways of conveying their meaning they develop a vocabulary more in keeping with a 'mathematical register' (Pimm, 1987), and one which helps to clarify their understanding. Students, however, have to have something worthwhile to talk about.

The use of non-routine questions to promote conversation

Non-routine questions that are 'open', or 'goal free' have been used to provoke reflective thought and develop higher order thinking in students so that their mathematical knowledge is deeper rather than shallow, and conceptual rather than procedural (Biggs, 1991; Ellerton & Clements, 1996). Non-routine questions may also have the virtue of being accessible to students at many levels. Open or investigative tasks often rely on students working in small groups to solve the problem. Such small group discussion can be an appropriate strategy to facilitate comprehension 'as students attempt to explore, investigate and solve problems together' (Ellerton & Clements, 1996). Although in the usual size classroom, careful monitoring of group discussions is difficult, and the subsequent development of mathematical understandings as well as communication skills cannot be guaranteed (Gooding & Stacey, 1993), the small group situation described in this article privileges the teacher and allows her to probe or guide students' thinking.

Two snapshots described in the following section illustrate the way in which the three aspects of the model — language arts techniques, linguistic structures of student understanding and conversation — come together.

The model explored

The model was developed for implementation with a small after-school tutor group conducted by a community organisation in a rural town. The examples of conversations presented here come from the first two-hour tutorial of the year. The students involved have chosen to attend the classes because they feel they lack the knowledge and confidence to 'get good marks' in the NSW Higher School Certificate. All the students report experiences of traditional mathematics classrooms, where they have focussed on memorising routines and procedures that can be applied to standard or stereotypical questions. The primary aim of the tutorials is to foster a more positive attitude to mathematics and hence to increase the students' confidence as well as their deep understandings (Biggs, 1991).

The introductory session serves to acquaint the students with the types of atypical tasks they may encounter in subsequent classes, and which also provide information about their mathematical strengths and weaknesses (Stoessiger & Edmunds, 1990). The students were presented with questions that addressed aspects of measurement, geometry, algebra, number and statistics. Two of the questions and the responses of the students will be discussed in this article. The conversation during the session was recorded as teacher notes.

There are two girls and one boy and the names used for them are pseudonyms. One student, Mary, is in Year 12, sitting the NSW Higher School Certificate in Mathematics (2U), a calculus-based course. The other students have just started Year 11. Ann is studying the General Mathematics course, which is best described as an algebra-based course, and Tom, the Mathematics Extension 1 course (old 3U course). The latter is an extension of the Mathematics (2U) or calculus-based course.

Snapshot 1: Square roots

The question presented to the students was:

What is 4^2 ? What is $\sqrt{81}$?

Describe in any way(s) you wish what you understand by the statement 'four squared'.

Describe in any way(s) you wish what you understand by the statement, 'find the square root of 81'.

Find the square root of 27. Record clearly and in as much detail as possible how you arrived at your answer.

(Adapted from *Principles and Standards for School Mathematics* (NCTM, 2000, p. 292).)

A calculator with only the four arithmetic operations was supplied.

The conversation began when the tutor (JF) noticed that Mary was busily plugging numbers into her calculator and asked her what she was doing.

1. M: Trying to find $\sqrt{27}$!
2. JF: Can you tell how the calculator can help?
3. M: I'm trying to find the answer.
4. JF: Are you recording your attempts? I would like to see exactly what you do... [Pause, while JF continues to observe Mary, who continues to plug numbers into the calculator according to her own, unarticulated logic, but she does write down the various calculator responses]. Mary, what is a square number?
5. M: A number which goes into 27 twice... no... a number timesed (sic) by itself.
6. JF: Do you have a solution?
7. T: [Confidently] It's three.

Ann and Mary at this point began to write down the 'answer' until Tom corrected himself, saying that three was the *cube* root of 27. After a longish pause, while Mary continued to consult her calculator, the conversation resumed.

8. M: There isn't one... It's got a long, long decimal point. I can't be bothered... It's point oh eight, eight over... so I've given up.

Tom, meanwhile had written:

$$\begin{aligned}
 x &= \sqrt{27} \text{ lies between } \sqrt{24} \text{ and } \sqrt{28} \\
 2\sqrt{6} &> x < 2\sqrt{7}, \quad x = \sqrt{27} \\
 5^2 &> x < 6^2
 \end{aligned}$$

9. JF: Can you write $\sqrt{27}$ another way? As a decimal?
10. T: I could if had a better calculator.
11. JF: That's the point of the question. Can you find it other than by pressing a button?
13. T: I can guess....Well, $\sqrt{25}$ is 5, so $\sqrt{27}$ is... 5.4. See, 5.42 is 26.6
14. JF: How did you do that?
15. T: $5^2 = 25$ and $0.4^2 = 1.6$, so $25 + 1.6 = 26.6$
16. JF: How can you check that?

Tom found that 5.4^2 was greater than 26.6 when he used the calculator. He commented on the fact that 5.4 was too big, but did not question the disparity with his long hand answer. JF chose not to address this problem at this time, but to return to it in the following lesson.

Some insights from the conversation

(Relevant line numbers in the conversation are in brackets.)

Generic/general understandings

Mary knows many mathematical rules, definitions or descriptions and contributes these readily (5). Her immediate response to the question, by

reciting a description of a square number, indicates a mathematical framework that consists of a collection of isolated facts. The recall of these facts is relied upon to provide insight to solve problems.

The definition of a square number appears to be a 'general' response (Bills & Grey, 2001), but Mary's frustration (8) at not being able to obtain a finite solution may indicate that her underlying concept of square numbers is more 'generic' (Bills & Grey, 2001). That is, Mary uses the set of numbers which are perfect squares as being an archetype for all square numbers.

This 'generic' understanding of square numbers is limiting and inhibits Mary's ability to recognise $\sqrt{27}$ as irrational, and to connect this with other, previously remembered characteristics of irrational numbers.

Tom's response to the question was possibly influenced by his recent classroom experience of surds. He recognised that $\sqrt{27}$ had to lie between $\sqrt{24}$ and $\sqrt{28}$ — two surds for which he could find simpler forms, but which did not seem to provide him with any further useful information (8). He recognised that $\sqrt{27}$ was not a perfect square, but his conclusion was that it could only be conveniently expressed as lying between two surds that he could in some way simplify. However, Tom did not try to simplify $\sqrt{27}$ as $3\sqrt{3}$, although he recognised it as the cube of 3(7). Tom's understanding of the simplification of surds may also be 'generic' rather than 'general' as he only attempted to simplify those surds that had a factor 4.

Mary could not accommodate Tom's approach at all, preferring to stay with the guess-and-check iteration to an 'answer', based on her understanding of square numbers. Tom, on the other hand was happy to accept the teacher's (JF) advice and look for a decimal approximation, although he was unhappy with the inadequate calculator (10). It would seem that these 'generic' understandings allow the students to act in a more or less procedural manner when confronted with routine, familiar situations that, for them, exemplify their understandings of square numbers, square roots, surds and irrational numbers.

Meaning of notation

As Tom continued to think, JF suggested he write down his thoughts. The use of incorrect notation revealed a limited understanding of the meanings of the inequality signs and how to read them successfully. Tom later revealed that that was the way he always wrote 'between' because x was between the two inequality signs (8). This idiosyncratic symbolic expression of 'between-ness' is an example of what Frid (1993) describes as students manipulating symbols without meaning, acting according to a set of ill-understood rules and procedures. When requested to read aloud the expression, beginning with 'x less than...' Tom's confusion was made apparent to him, and he volunteered the appropriate correction.

Understanding of the process of squaring numbers

Tom's concept of $\sqrt{27}$ lying between 5 and 6 began to develop as he struggled with the idea. Even when he placed $\sqrt{27}$ somewhere between 5 and 6, using the reference points of 25 and 36, he demonstrated no 'feel' for a possible

decimal approximation as needing to be nearer to 5 than to 6. His expansion of 5.42^2 to 26.6 provided a confirmation of his 'guess' and another misconception to address (15).

The error in Tom's thinking was made apparent by requiring the student to use a calculator that restricted the type of operations available. Tom was confident to carry out the calculation by hand, and in so doing, revealed his misunderstanding. However, he did not pause to wonder at the discrepancy between his pen-and-paper calculation and the calculator display. Tom's focus was on obtaining a unique answer and he could not see the inconsistency between the two answers.

Students meet irrational roots in two contexts. These are the algebraic context, where surds are simplified and manipulated algebraically, and in arithmetic problems such as finding the radius of a circle, given the area. In this latter instance, recourse to a calculator and the subsequent decimal approximation provides the answer. These two contexts are often dissociated from each other and the connections between them often not made explicit. Students may therefore remain with a 'generic' idea of squares and square roots because they lack the experiences which would enable them to generalise the concepts.

Snapshot 2: Rectangles

The question posed to the students was:

Consider rectangles with a fixed area of 36 square units. What are some possible rectangles? How many different rectangles can be made? Justify your answers in as much depth and detail as possible. (Adapted from *Principles and Standards for School Mathematics* (NCTM, 2000, p. 228).)

The conversation focussed on the students' geometrical understandings although the task was designed to elicit a measurement and/or an algebraic response. This conversation was more free-ranging than the former because the three students remained engaged and felt they could contribute. JF began by asking Ann for her responses to the above question. Ann stated that she did not understand it.

Mary tried to clarify the problem:

16. M: The angles are acute, obtuse...
17. T: It's about rectangles.
18. M: [After drawing several quadrilaterals, none of them rectangles] It doesn't make sense.
19. JF: What is a rectangle?
20. M: A four sided shape.
21. T: It's got four sides, four angles and two different length sides.
22. JF: What is a trapezium?
23. M: It looks like... it has three different length sides.

JF Interpreted this statement by drawing an isosceles trapezium, then presented Mary with several other trapezia in different orientations, then redirected her attention to Tom and Ann. Ann was asked to read the question and she suggested one possible rectangle. This was sufficient for Mary to write down other rectangles having whole number dimensions. Tom agreed with Mary.

24. JF: Any others?
25. T: You can't have 6 x 6. It's not a rectangle!
26. JF: Isn't it?
27. M, A & T: [Not quite in unison] No. It's got all even sides. It's a square.
A square isn't a rectangle.
28. JF: Tell me, what are the characteristics of a rectangle?

Tom repeated his earlier statement. Mary added the fact that opposite sides were equal and Ann stated that all angles were right angles. JF meanwhile had cut out a square and as the students listed the properties, pointed them out on the square. Silence. The image of a unique figure with all equal sides dominated and confused the students. Tom was openly unconvinced.

29. JF: What other quadrilaterals are there?
30. T: Rhombus, trapezium, parallelogram
31. JF: Is a rectangle a parallelogram?
32. T: Yes, but it has four right angles.
33. JF: It does have four, but you really only need one to make a parallelogram a rectangle.
34. A: How come?

JF attempted a rough demonstration asking the students to form a 'parallelogram' with their forearms, then, to make a right angle at one vertex whilst keeping their arms parallel. Mary managed this successfully, but Tom and Ann focused only on making the right angle. Mary demonstrated, they mimicked her, but seemed unable to make any convincing connections between the ideas.

Some Insights from the conversation

(Relevant line numbers in the conversation are in brackets.)

A 'generic' picture of quadrilaterals

All three of the students could identify some properties of a rectangle, but not those that necessarily define a rectangle as distinct from other quadrilaterals (20–27). For example, Tom lists the properties, 'four sides, four angles and two different length sides' (21) and Mary simply states, 'a four-sided shape' (20). These phrases suggest that the students are using the concept of a rectangle as a 'generic' example of the group of quadrilaterals (Bills & Grey, 2001). Tom and Mary could provide a list of some properties particular to rectangles, such as the figure having four right angles and equal opposite

sides, but their responses did not indicate that they saw any relationship between one property and another. It was as if the students were imagining a rectangle and then listing the 'visible' properties.

All students initially denied the fact that a square was a rectangle (25–27). The students used the simile 'A square is like a rectangle', rather than the statement 'A square is a rectangle'. Tom and Ann did agree that the square was like a rectangle (not recorded here as verbatim), but, because of its having all sides equal, it could not be a rectangle. The students struggled with the abstract notion of the set of squares being a subset of the set of rectangles. It is as if, as suggested above, the students were listing properties as they 'saw' the figure. Dominating their reasoning was the 'visible' fact of the square having all four sides equal, emphasised by the paper model (28), not two 'different length sides' which would be consistent with their image of a rectangle. Consequently, the students perceived and so conceived a square as being quite distinct from a rectangle.

Tom was prepared to agree that a rectangle could be a parallelogram, although, this was qualified by his 'Yes, but it has four right angles'. Tom's use of 'but' (32) indicates some cognitive reservations about a rectangle actually being a parallelogram. Had he used 'and' ('Yes, and it has four right angles'), one could infer that Tom conceptualised a rectangle as a figure possessing all the properties of a parallelogram as well as other particular properties.

This examination of the students' use of language leads to an hypothesis that the students see (literally) the rectangle as representative of all quadrilaterals. This mental representation of quadrilaterals is 'generic' (Bills & Grey, 2001). If the students could conceive of a 'generalised' (Bills & Grey, 2001) quadrilateral they could then perceive that different quadrilaterals share some common properties and yet also possess other, particular properties. This perception underpins the conception that rectangles are a particular instance of quadrilaterals. The same logic would then compel the students to the conclusion that a square is, indeed a rectangle. In other words, the students need to develop skills and concepts that enable them to reason abstractly about quadrilaterals. In some way, this conversation began that process.

Conclusions

Ann, Tom and Mary should have had considerable experience of geometry and some exposure to irrational numbers in the Years 7 to 10 (Board of Studies NSW, 1996a; 1996b). Examples of activities and questions suggested in the syllabus would have provided learning experiences to develop the understandings discussed in this article.

Much more could be written about these conversational fragments. Other cognitive models could be used as a basis for describing the students' levels of understanding of the concepts they were dealing with. The purpose of this article, however, is to present an example of a possible model that allows a teacher to uncover some of the hidden understandings (or misunderstand-

ings) of students. The direction of the conversations was determined by what the students were doing and saying, although the written version cannot convey the dynamics of the conversations. However, the above examples demonstrate how students' understandings of square roots and of quadrilaterals may be inferred from what they say and how they say it. Some conclusions are speculative, but serve to demonstrate how a teaching and learning model based on the learning of mathematics as a language could be developed to identify and codify students' strengths and weaknesses.

In most classrooms, teachers cannot always focus so carefully on the conversation between students. This is the difficulty with group work highlighted by Gooding and Stacey (1993). Even when teachers do listen intently to what students say, they often do so in order to correct students, rather than dwell on how student errors reveal possible student cognitive development. This second goal is usually more easily carried out by analysis of written work. This too, may be problematic when students experience difficulty accessing the meaning of formal mathematical symbols and writing. Consequently, students may not be able to reveal their thinking, such as, for example, Tom's idiosyncratic representation of $5 < \sqrt{27} < 6$ (8). By the time a student has reached the senior years of high school teachers rarely think to look for these misunderstandings in the daily classroom, or have the time to delve into the causes of them. In this particular group, however, there was opportunity to encourage, monitor and guide student talk, and to note aspects that might have gone unnoticed in a normal classroom.

The small group, the lack of curriculum constraints, the informal setting and the focus on students articulating their thoughts and challenging each other, with the group conversations able to be monitored by the tutor, is a model that may serve to guide teachers helping struggling students. In particular, it provides for the creation of an environment where teachers can focus on the language constructs of students to expose student understandings, and misunderstandings, in ways that student writing cannot.

References

- Barnes, M. (1999). Cumulative and exploratory talk in a collaborative learning classroom. In J. Truran & K. Truran (Eds), *Making the Difference* (Proceedings of the twenty-second annual conference of the Mathematics Education Research Group of Australia, pp. 53–59). Adelaide: MERGA.
- Bickmore-Brand, J. (1993). Applying language-learning principles to mathematics teaching. In M. Stephens, A. Waywood, D. Clarke & J. IZard (Eds), *Communicating Mathematics: Perspectives from classroom practice and current research* (pp. 79–90). Melbourne: ACER.
- Bell, G. (1993). Mathematics x language = language x mathematics. In M. Stephens, A. Waywood, D. Clarke & J. IZard (Eds), *Communicating Mathematics: Perspectives from classroom practice and current research* (pp. 140–152). Melbourne: ACER.
- Biggs, J. (1991). Enhancing learning in the context of school. In R. Mulcahy, R. Short & J. Andrews (Eds), *Enhancing Learning and Thinking* (pp. 35–52). NY: Praeger.
- Bills, C & Grey, E. (2001). The 'particular', 'generic' and 'general' in young children's mental calculations. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th Annual Conference of the International Group for the Psychology of Mathematics Education Vol. 2* (pp. 153–160). Utrecht: PME.
- Board of Studies NSW (1996a). *Mathematics Syllabus Years 9–10: Advanced Course, Stage 5*. Sydney: Author.
- Board of Studies NSW (1996b). *Mathematics Syllabus Year 9–10: Intermediate Course, Stage 5*. Sydney: Author.
- Ellerton, N. & Clements, M. (1996). Researching language factors in mathematics education: The Australian contribution. In W. Atweh, K. Owens & P. Sullivan (Eds), *Review of Mathematics Education in Australasia 1992–1995* (pp. 191–235). Sydney: MERGA.
- Frid, S. (1993). Communicating mathematics: A social sharing of language and decisions pertaining to truth and validity. In M. Stephens, A. Waywood, D. Clarke, & J. IZard (Eds), *Communicating Mathematics: Perspectives from classroom practice and current research* (pp. 26–40). Melbourne: ACER.
- Gooding, A. & Stacey, K. (1993). How children help each other learn in groups. In M. Stephens, A. Waywood, D. Clarke, & J. IZard (Eds), *Communicating Mathematics: Perspectives from classroom practice and current research* (pp. 41–50). Melbourne: ACER.
- McGregor, M. (1993). Interaction of language competence and mathematics learning. In M. Stephens, A. Waywood, D. Clarke, & J. IZard (Eds), *Communicating Mathematics: Perspectives from classroom practice and current research* (pp. 51–59). Melbourne: ACER.
- National Council of Teachers of Mathematics (2001). *Principles and Standards for School Mathematics*. Reston: NCTM.
- Pimm, D. (1987). *Speaking Mathematically: Communication in Mathematics Classrooms*. London: Routledge & Kegan Paul.
- Stoessiger, R. & Edmunds, J. (1990). The role of challenges. In Bickmore-Brand, J. (Ed.), *Language in Mathematics* (pp. 109–116). Melbourne: Australian Reading Association.